ISOMORPHISM GROUPS OF METABELIAN GROUPS GENERATED BY FOUR INDEPENDENT OPERATORS OF ORDER p

BY

MARY ANICE SEYBOLD

A.B., Illinois College, 1929

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SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN MATHEMATICS IN THE GRADUATE SCHOOL OF THE UNIVERSITY OF ILLINOIS, 1947

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ACKNOWLEDGMENT

The author wishes to express sincerest thanks to Professor H. R. Brahana for his skillful guidance in the preparation of this thesis.

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A metabelian group is a group whose central contains its commutator subgroup. We shall restrict our consideration to metabelian groups which satisfy the following conditions:

(1) The commutator subgroup of the metabelian group coincides with the central.

(2) The operators of the metabelian group are all, except identity, of order p, a prime not equal to 2.

(3) The metabelian group is generated by four independent operators.

We make restriction (1) because whenever the central does not coincide with the commutator subgroup, then the central is the direct product of the commutator subgroup and an abelian group of type 1, 1, 1, ...; and the metabelian group itself is the direct product of this same abelian group and a metabelian group whose central coincides with its commutator subgroup.[1]

We make restriction (2) because the study of such groups is essential to the fundamental problem of classification of metabelian groups. The question of whether or not two metabelian groups are isomorphic leads to the question of whether or not two groups of this type are isomorphic. Brahana has pointed out [2] that a metabelian group generated by elements having periods that are powers of a prime has a characteristic quotient group whose operators are all, except identity, of order p. This quotient group is determined by the invariant subgroup generated by all elements of the metabelian group that are pth powers of elements of the group.

Restriction (3) is made to limit the scope of the work. The case for three independent generators is of little interest since in metabelian groups generated by three independent operators all the elements of the commutator subgroup are commutators.

Four independent generators taken in pairs determine six commutators. If all the six commutators are independent, the group is said to be the master group. All other groups satisfying condi-

[1] Brahana, H. R., "Finite Metabelian Groups and Plücker Line-Coordinates," *American Journal of Mathematics*, Vol. LXII, p. 365.

[2] Brahana, H. R., "Metabelian Groups and Trilinear Forms," *Duke Mathematical Journal*, Vol. I, p. 185.

tions (1), (2) and (3) are quotient groups of this one with respect to invariant subgroups of order p, p^2 or p^3 in the central. Brahana has shown[3] that there are twelve metabelian groups, to one and only one of which any group satisfying conditions (1), (2) and (3) is isomorphic. The eleven quotient groups of the master group are classified according to the representation in an S_5 of the corresponding invariant subgroup in the central of the master group.

A cyclic subgroup consisting entirely of commutators corresponds to a point in S_5 on the four dimensional variety

$$V_4 : p_1p_6 - p_2p_5 + p_3p_4 = 0.$$

A cyclic subgroup generated by a noncommutator of the central corresponds to a point in S_5 not on V_4.

Subgroups of orders p^2 and p^3 in the central of the master group correspond in S_5 to lines and planes whose relations to V_4 may be used to classify the corresponding quotient groups.

Geometric Representation of the Master Group. In this thesis, a geometric representation of the master group is set up in a projective finite S_3, modulus p. Cyclic subgroups G/C correspond to points. Subgroups of order p in C which consist entirely of commutators correspond to proper lines. Noncommutators of the central generate subgroups of order p which correspond to improper lines. A correspondence is established between linear transformations in this S_3 and certain isomorphisms of the group by means of the matrix representation of isomorphisms and transformations.

Isomorphism Group of the Master Group. Let a nonsingular point transformation in S_3 be represented by the matrix (a_{ij}) where i, j, $= 1, 2, 3, 4$. Let the induced line transformation be represented by the matrix (c_{mn}), where m, n $= 1, \ldots, 6$. Then the 10×10 matrix

$$A = \left(\begin{array}{c|c} a_{ij} & 0 \\ \hline 0 & c_{mn} \end{array}\right) \qquad \begin{array}{l} i, j = 1, 2, 3, 4 \\ m, n = 1, \ldots, 6 \end{array}$$

represents not only a transformation in S_3 but also an isomorphism of G. The matrix (a_{ij}) completely determines the isomorphism of G. The matrix (c_{mn}) represents the corresponding induced isomorphism of the central, and is incorporated into the 10×10

[3] Brahana, H. R., "Finite Metabelian Groups and Plücker Line-Coordinates," *American Journal of Mathematics*, Vol. LXII, p. 365.

matrix merely for convenience. The isomorphisms and transformations with matrices A form nonabelian groups.

Isomorphisms of G with matrices of the form

$$B = \left(\begin{array}{c|c} 1 & b_{kl} \\ \hline 0 & 1 \end{array}\right) \qquad \begin{array}{l} k = 1, 2, 3, 4 \\ l = 1, \ldots, 6 \end{array}$$

form an abelian group, invariant under the group of isomorphisms with matrices A. The isomorphism group J of the master group is the product of isomorphism groups with matrices A and B.

Geometric Representations of Invariant Subgroups. Geometric representations of invariant subgroups of orders p, p^2 and p^3 in the central of the master group are constructed in a projective finite S_3', modulus p. A subgroup of order p consisting entirely of commutators is represented by a special complex. A special complex consists of all the lines that intersect a proper line, called the axis of the complex.

A cyclic subgroup of order p generated by a noncommutator of the central is represented by a general complex. A general complex consists of all the lines that satisfy the condition for intersection with an improper line, which we call the axis of the complex.

The four kinds of subgroups of order p^2 in the central correspond to the four kinds of pencils of complexes. The axes of the complexes of a pencil form a family of lines linearly dependent on any two of the axes. Such a family of lines is called a pencil of lines. It may contain 0, 1, 2 or $p + 1$ proper lines according as the pencil of complexes contains 0, 1, 2 or $p + 1$ special complexes. The line in S_5 corresponding to the pencil of complexes correspondingly does not cut V_4, is tangent to V_4, cuts V_4 in two points or lies entirely in V_4.

The five kinds of subgroups of order p^3 in the central of the master group correspond to the five kinds of bundles of complexes. A bundle of complexes may have 1, $p + 1$, $p + 1$, $2p + 1$ or $p^2 + p + 1$ special complexes according as the corresponding plane in S_5 cuts V_4 in a point, in a proper conic, in a line, or in two intersecting lines; or lies on V_4.

Similar representations exist for subgroups of higher order in the central of the master group, but they correspond to quotient groups that can be generated by fewer than four independent operators.

Geometric Representation of Quotient Groups. Geometric representations of the eleven quotient groups are established in a projective finite S_3'', modulus p. Subgroups of order p of the central quotient group of one of these eleven groups are mapped onto the points of S_3''. Subgroups of order p of the central are mapped onto sets of lines. A set consists of all lines in S_3'' having the same coordinates as the lines in S_3 representing the subgroups of the central of the master group which correspond to the given subgroup of the quotient group.

If S_3' is mapped onto S_3'' by correspondence of points with the same coordinates, the geometric representations of the quotient group and its corresponding invariant subgroup overlap. By means of this overlapping it is possible to prove the theorem: *The transformations that leave fixed the geometric representation of a given quotient group are the transformations that leave fixed the geometric representation of the corresponding invariant subgroup of the master group.*

Isomorphism Groups of Quotient Groups. By means of the matrix representation of transformations and isomorphisms it is established that isomorphisms of a given quotient group are induced by and only by the isomorphisms of the master group which leave fixed the corresponding invariant subgroup of the master group. A method is given for changing the 10×10 matrix of an isomorphism of the master group which leaves fixed an invariant subgroup of the central into a smaller matrix of an isomorphism of the corresponding quotient group, by applying the defining relations of the quotient group to the 10×10 matrix and then deleting the unnecessary rows and columns.

As in the master group, we can exhibit the isomorphism group of any given quotient group as the product of two groups, one of which is invariant under the entire group of isomorphisms of the given quotient group.

VITA

Mary Anice Seybold was born January 29, 1908, at Baylis, Illinois. She was graduated from Perry Community High School in 1925. She received the A.B. Degree with Special Honors in Mathematics from Illinois College in 1929, the A.M. from the University of Illinois in 1931. She was chairman of the Illinois Section of the National Council of Teachers of Mathematics in 1941. She has taught mathematics in the high schools of Monticello and Freeport, at Illinois College and the University of Illinois; and is now Head of the Mathematics Department at North Central College at Naperville, Illinois.

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